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On Transitivity of Proximinality

V. INDUMATHI

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802 U.S.A.; and Faculty of Electronics, Military College of EME, Trimulgherry, Secunderabad – 500 015, India

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A characterization of normed linear spaces, which "transmit" proximinality for subspaces of finite codimension, is given. This also gives a solution to a problem of Pollul [10].

1. NOTATIONS

Throughout the paper E stands for a real normed linear space, E^* for its dual, and M will always denote a closed linear subspace of E.

If M is a subspace of E, we write $M^\perp$ for its annihilator

$$\{f \in E^*: f(m) = 0, \text{ for all } m \in M\}$$

in E^*. The space $M^\perp$ is isometrically isomorphic to $(E/M)^*$ through the canonical linear isometry, and so if the codimension of M is finite, say n, then the dimension of $M^\perp$ is also n.

For any normed linear space E, we will denote by E_1 the unit ball of E. For instance, the unit ball of the space $(M^\perp)^*$ will be denoted by $(M^\perp)_1^*$.

For $x \in E$, we denote by $\hat{x}$ the image of x in the canonical embedding of E into E^{**} and by $\theta_M(x)$ the restriction of $\hat{x}$ to the subspace $M^\perp$ of E^*.

If C is a bounded, closed, convex subset of a normed linear space then $S(C)$ will denote the collection of linear functionals in E^* which attain their supremum on C. That is,

$$S(C) = \{f \in E^*: \exists x \in C, \text{ such that } f(x) = \sup_{y \in C} f(y)\}.$$

In particular, if $C = E_1$, then $S(E_1)$ denotes the set of functionals in E^* which attain their norm on E.

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In the following, $C(Q)$ denotes the Banach space, with the uniform norm, of real valued continuous functions defined on the compact, Hausdorff space Q, and $L_1(T, \nu)$ denotes the space of real valued Lebesgue integrable functions on the positive measure space (T, ν) endowed with the norm $\|x\| = \int_T |x| \, d\nu$. We recall that $C(Q)^*$, the dual of $C(Q)$, is $M(Q)$, the space of regular Borel measures on Q. Also, we assume that the measure space (T, ν) is such that $L_1(T, \nu)^*$, the dual of $L_1(T, \nu)$, can be identified with the space of real valued, essentially bounded functions defined on the positive measure space (T, ν). (This will be the case, e.g., if (T, ν) is σ-finite.)

If $\mu \in C(Q)^*$, then $\mu = \mu^+ - \mu^-$ will denote the Jordan decomposition of the measure and $\operatorname{supp}(\mu)$ its support, the complement of the largest open subset U of Q such that $|\mu|(U) = 0$.

If M is a subspace of a normed linear space E and $x \in E$, we define

$$P_M(x) = \{m_0 \in M : \|x - m_0\| = \inf_{m \in M} \|x - m\|\}.$$

The subspace M is called *proximinal* if the set $P_M(x)$ is nonempty for each $x \in E$.

If T is a map, we denote by T^{-1} the inverse map. For instance, for $m \in M$, $P_M^{-1}(m)$ would denote the set of all elements x in E for which $m \in P_M(x)$. Also, if T is defined on a normed linear space E and M is any subspace of E, $T|_M$ would denote the restriction of the map T to the subspace M.

Finally, if A and B are subsets of a normed linear space E, $A \backslash B$ will denote the complement of B in A.

All other undefined notation or terminology is standard and can be found in [4].

2. The $R(n, m)$ Space

We start this section with the following results of Garkavi regarding proximinal subspaces of finite codimension.

THEOREM A (Garkavi [5]). *Let M be a subspace of finite codimension in a normed linear space E. Then M is proximinal if and only if*

$$\theta_M(E_1) = (M^\perp)_1^*.$$

THEOREM B (Garkavi [7]). *Let M be a subspace of finite codimension in $C(Q)$. Then M is proximinal if and only if the annihilator space satisfies the following three conditions:*

(i) $\operatorname{supp}(\mu^+) \cap \operatorname{supp}(\mu^-) = \phi$, *for every* $\mu \in M^\perp \setminus \{0\}$.

(ii) μ_2 *is absolutely continuous with respect to* μ_1 *on* $\operatorname{supp}(\mu_1)$, *for every pair* $\mu_1, \mu_2 \in M^\perp \setminus \{0\}$.

(iii) $\operatorname{supp}(\mu_2) \setminus \operatorname{supp}(\mu_1)$ *is closed for each* $\mu_1, \mu_2 \in M^\perp \setminus \{0\}$.

Theorem A implies that if M is of finite codimension n in any normed linear space E, then

$$M \text{ is proximinal} \Leftrightarrow \text{Every subspace } N \supseteq M \text{ with codim } N \leqslant n \text{ is proximinal.} \quad (1)$$

Theorem B implies that if $E = C(Q)$ and M is a subspace of finite codimension in E, then

$$M \text{ is proximinal} \Leftrightarrow \text{Every subspace } N \supseteq M \text{ such that codim } N = 2 \text{ is proximinal.} \quad (2)$$

Also, if $E = c_0$, the space of sequences of real scalars converging to zero with the sup norm (Pollul [10], Blatter and Cheney [1]), or if E is an (incomplete) inner product space (Deutsch [3]), we have for a subspace M of finite codimension in E, then

$$M \text{ is proximinal} \Leftrightarrow \text{Every subspace } N \supseteq M \text{ such that codim } N = 1 \text{ is proximinal. Or equivalently every hyperplane containing } M \text{ is proximinal.} \quad (3)$$

However, this special behaviour of $C(Q)$ or c_0 is not typical. In fact, if E is any infinite dimensional $L_1(T, \nu)$ space, one can construct a subspace M of finite codimension n ($n \geqslant 2$) such that every subspace N of E with $N \supset M$ and $N \neq M$ is proximinal, while M itself is not proximinal [8]. Thus in view of (1)–(3) and the above-mentioned behaviour of $L_1(T, \nu)$, it seems natural to make the following definition.

DEFINITION 1. A nonreflexive space E is said to be a $R(n, m)$ *space* (m and n positive integers, $n \geqslant m$) if M is a subspace of finite codimension k in E, $m \leqslant k \leqslant n$, then

$$M \text{ is proximinal} \Leftrightarrow \text{Every subspace } N \supseteq M \text{ such that codim } N = m \text{ is proximinal.} \quad (4)$$

Remark 1. It would be preferable to have the following simpler definition in place of definition 1, if the two were equivalent.

A nonreflexive space E is a $R(n, m)$ space ($n > m \geqslant 1$) if M is a subspace of codimension n in E, then (4) holds.

Clearly, Definition 1 implies the above. However, the above definition would imply Definition 1 only if every subspace M of codimension k in E ($m \leqslant k \leqslant n$), with the property that all subspaces N, $N \supseteq M$, and codim $N = m$ are proximinal, contains some subspace L of codimension n in E with the same property. That is, L is such that every subspace $N \supseteq L$ with codim $N = m$ is proximinal. If we consider the particular case where $m = 1$, this would mean that every k-dimensional subspace of $S(E_1)$ should be contained in some n-dimensional subspace of $S(E_1)$. Even for this simplest case we do not know whether the above-mentioned condition holds in all normed linear spaces.

The space c_0 and the incomplete inner product spaces are $R(n, 1)$ spaces for $n \geqslant 1$, $C(Q)$ is a $R(n, 2)$ space for $n \geqslant 2$, and if n is any positive integer, every infinite dimensional $L_1(T, v)$ space is *not* a $R(n, m)$ space for any $n > 1$ and with $m < n$.

It is worth mentioning here that any infinite dimensional $C(Q)$ space is not a $R(2, 1)$ space and hence not a $R(n, 1)$ space for $n \geqslant 2$. The following example, adapted from Phelps [9], shows that one can construct a subspace M of codimension 2 in any infinite dimensional $C(Q)$ space such that every hyperplane containing M is proximinal but M itself is not proximinal.

Select a sequence $(q_n)_{n=1}^{\infty}$ in Q, with $q_n \neq q_m$ for $n \neq m$, which has a cluster point $q_0 \in Q$, with $q_0 \neq q_n$ for $n = 1, 2....$ Define $\mu_1, \mu_2 \in M(Q)$ by

$$\mu_1 = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{q_n} + \delta_{q_0},$$

$$\mu_2 = \sum_{n=1}^{\infty} \frac{1}{4^n} \delta_{q_n},$$

where

$$\delta_q(B) = \begin{cases} 1 & \text{if } \; q \in B \\ 0 & \text{if } \; q \notin B \end{cases}, \quad \text{for any subset } B \text{ of } Q.$$

Let $M = \{x \in C(Q): \mu_i(x) = 0, \text{ for } i = 1, 2\}$. Then $M^{\perp}$ is the two dimensional subspace of $M(Q)$ generated by μ_1 and μ_2. If a is any scalar, we have $(\mu_1 + a\mu_2)(q_n) = (1/2^n) + (a/4^n) > 0$ if $2^n > -a$. This implies that for any $\mu \in M^{\perp}$, we have $\operatorname{supp}(\mu^+) \cap \operatorname{supp}(\mu^-) = \phi$ or, equivalently, every $\mu \in M^{\perp}$ attains its norm on $C(Q)$. Hence every hyperplane containing M is proximinal. However, M is not proximinal in $C(Q)$ since condition (ii) of Theeorem B does not hold for μ_1 and μ_2.

In the sequel when we say that a normed linear space is a $R(n, m)$ space, we will presume that E has at least one proximinal subspace of codimension n. This, in turn, would imply that the set $S(E_1)$ has at least one n-

dimensional subspace contained in it. We note that $C(Q)$ and $L_1(T, \nu)$ have proximinal subspaces of finite codimension n, for every n (Garkavi [6, 7]).

Given $m \geqslant 1$, if a normed linear space E is a $R(n, m)$ space for all $n \geqslant m$, then we call E a $R(m)$ space. Thus c_0 would be a $R(1)$ space, $C(Q)$ would be a $R(2)$ (but not a $R(2, 1)$) space. Below we give a simple but useful comment (Corollary 1) regarding $R(1)$ spaces. We need the following result of Garkavi and Proposition 1 in the sequel.

THEOREM C (Garkavi [5]). *Let E be a normed linear space and M be a subspace of E. Then M is proximinal in E if and only if for every $x \in E$, there exists an $y \in E$ such that*

$$f(x) = f(y), \qquad \text{for every } f \in M^{\perp},$$

and

$$\|y\| = \|\hat{x} \mid M^{\perp}\|.$$

PROPOSITION 1. *Let E be a normed linear space with E^* smooth at every point of $S(E_1) \cap E_1^*$. If M is a subspace of E with $M^{\perp} \subseteq S(E_1)$, then M is proximinal in E.*

Proof. We would show that under the assumptions of Proposition 1, the conditions of Theorem C hold.

Let $x \in E$. Define $\Phi = \hat{x} \mid M^{\perp}$. Then $\Phi \in (M^{\perp})^*$. Let $\Phi_0 \in E^{**}$ be a norm preserving extension of Φ. Then $\|\Phi_0\| = \|\Phi\| = \|\hat{x} \mid M^{\perp}\|$.

Select a sequence (fn) in $(M^{\perp})_1$ such that $fn(x) \to \|\hat{x} \mid M^{\perp}\|$. Let f be a w^* limit of (f_n). Since $M^{\perp}$ is w^* closed, we have $f \in M^{\perp}$. Also $f(x) = \|\hat{x} \mid M^{\perp}\|$ and so $\|f\| = 1$.

We have $M^{\perp} \subseteq S(E_1)$ and hence there exist an $y \in E_1$ such that $f(y) = \|y\|$. The functionals Φ_0 and $\hat{y}$ in E^{**} support $f \in S(E_1) \cap E_1^*$. Since E^* is smooth at f, we have $\Phi_0 = \hat{y}$. This implies that

$$\Phi_0(f) = f(x) = f(y), \qquad \text{for every } f \in M^{\perp},$$

and

$$\|\Phi_0\| = \|\hat{x} \mid M^{\perp}\| = \|y\|.$$

We have the following easy consequences of the proposition.

COROLLARY 1. *Let E be a normed linear space with E^* smooth at every point of $S(E_1) \cap E_1^*$. Then E is a $R(1)$ space.*

COROLLARY 2. *If E^* is smooth then E is a $R(1)$ space. In particular, if E is an (incomplete) inner product space, then E is a $R(1)$ space.*

If E is a weakly locally uniformly rotund (WLUR) space then E^* is smooth at every point in $S(E_1) \cap E_1^*$. Yorke [13]). If E is a separable quasi reflexive Banach space of deficiency one, then one can define an equivalent norm on E such that E^{**} is smooth. (Smith [12]). Thus we have

COROLLARY 3. *If E is WLUR, then E is a $R(1)$ space.*

COROLLARY 4. *If J denotes the James space, there exists an equivalent norm on J such that J^* is a $R(1)$ space as under the new norm.*

The converse of Corollary 1 is not true, c_0 being the counterexample. For the sake of completeness, we give below a proof (Lemma 3) for the fact that c_0 is a $R(1)$ space. Also, this proof is simpler than the ones given in [1] and [10].

If $x_1, x_2, ..., x_n$ are elements in a normed linear space E, let $[x_1, x_2, ..., x_n]$ denote the subspace generated by the elements $x_1, x_2, ..., x_n$. Then we have

LEMMA 2. *Let E be a normed linear space with a monotone basis $(e_i)_{i=1}^{\infty}$. Let $(e_i^*)_{i=1}^{\infty}$ denote the corresponding biorthogonal functionals in E^*. If M is a subspace of finite codimension in E such that $M^{\perp} \subseteq [e_1^*, e_2^*, ..., e_k^*]$, for some finite k, then M is proximinal.*

Proof. Let M_k denote the closed subspace generated by the infinite set of elements $e_{k+1}, e_{k+2}, ...,$ and Q denote the natural projection from E onto M_k. Let $m = \sum_{i=k+1}^{\infty} m_i e_i$ and $x = \sum_{i=1}^{\infty} x_i e_i$ (where m_i and x_i are scalars) denote arbitrary elements in M_k and $E \backslash M_k$. Then we have

$$\begin{aligned} \|x - m\| &= \left\| \sum_{i=1}^{k} x_i e_i + \sum_{i=k+1}^{\infty} (x_i - m_i) e_i \right\| \\ &\geqslant \left\| \sum_{i=1}^{k} x_i e_i \right\| = \|x - Qx\| \end{aligned}$$

and $Qx \in P_{M_k}(x)$. Thus M_k is proximinal in E. Then, by (1), M is also proximinal in E.

LEMMA 3. *c_0 is a $R(1)$ space.*

Proof. Let M be a subspace of finite codimension in c_0 such that $M^{\perp} \subset S(E_1)$. We have to show that M is proximinal.

Since $M^{\perp} \subset S(E_1)$ and $M^{\perp}$ is finite dimensional, there is a positive integer k such that $M^{\perp} \subseteq [e_1^*, ..., e_k^*]$, where $(e_i^*)_{i=1}^{\infty}$ is the natural basis of $l_1 = (c_0)^*$. Thus, by Lemma 2, M is proximinal in c_0.

3. The Problem of Pollul

In the sequel, we will use the following notation of Pollul [10]. If M is a subspace of any normed linear space E, then $M \overset{(p)}{\subset} E$ will mean that M is a proximinal subspace of E.

Problem (Pollul [10]). Which nonreflexive spaces E satisfy

$$G \overset{(p)}{\subset} M, \qquad M \overset{(p)}{\subset} E, \qquad \dim M/G = \dim E/M = 1 \Rightarrow G \overset{(p)}{\subset} E?$$

Definition 2. A normed linear space E is said to be a $P(n)$ *space* $(n \geqslant 2)$ if

$$G \overset{(p)}{\subset} M, \qquad M \overset{(p)}{\subset} E, \qquad \dim E/G \leqslant n \Rightarrow G \overset{(p)}{\subset} E.$$

A normed linear space will be called "*Pollul space*," abbreviated as P *space*, if it is a $P(n)$ space for every $n \geqslant 2$. That is, proximinality is transitive.

We now restate the problem of Pollul in a more general form and proceed to give an answer to it.

Problem 1. Which nonreflexive spaces are $P(n)$ $(n \geqslant 2)$ spaces?

We note that when $n = 2$, the above problem reduces to that of Pollul. Before we give a characterization of $P(n)$ spaces, we need the following definitions and observations.

Definition 3. Let E be a normed linear space. Let f and g be elements of the dual space E^*. We say f is *strongly orthogonal* to g if f attains its norm on E at a point in the kernel of g or equivalently if there exists an $x \in E_1 \cap g^{-1}(0)$ such that $f(x) = \|x\|$.

We observe that if f is strongly orthogonal to g, then

$$\| f \| = \| f \mid g_{(0)}^{-1} \| = \inf_{\alpha} \| f - \alpha g \|,$$

and therefore $f \in P_{[g]}^{-1}(0)$. Hence f is orthogonal to g (in the sense of Birkhoff).

The converse does not hold. If we consider f and g in $l_\infty = (l_1)^*$ given by

$$f = \left(1, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \dots, 1 - \frac{1}{n}, \dots\right)$$

and

$$g = (1, 0, 0, \dots\dots),$$

then $\|f\| = 1 = \|f \mid g_{(0)}^{-1}\|$, but f does not attain its norm at any point in $g^{-1}(0)$.

If F is a finite dimensional subspace of E^* given by $F = [f_1, f_2, ..., f_n]$, where $f_i \in E^*$, for $1 \leqslant i \leqslant n$, and $f \in E^* \backslash F$, we say that f is *strongly orthogonal to the subspace* F if it attains its norm on E_1 at a point $x \in \bigcap_{i=1}^{n} f_i^{-1}(0)$.

DEFINITION 4. Let E be a normed linear space and F be a subset of the dual space E^*. Then F is called *orthogonally linear* if

$$f \in F, \qquad g \in F, \qquad \text{and} \qquad f \text{ is strongly orthogonal to } g \Rightarrow [f, g] \subseteq F.$$

We observe that if $E = c_0$ or an inner product space [3], then $S(E_1)$ is a linear subspace of E^* and hence orthogonally linear. However, it is easy to construct examples of infinite dimensional $C(Q)$ or $L_1(T, \nu)$ spaces for which the set of norm attaining functionals is not orthogonally linear. For example, if $E = l_1$, then $S(E_1)$ is not orthogonally linear.

It follows from Theorem A of Garkavi that if M is a subspace of finite codimension in any normed linear space E, then M is proximinal implies that every $f \in M^{\perp}$ is in $S(E_1)$. In the particular case when M is a hyperplane in E, this is also a sufficient condition for proximinality of M.

Let E be a normed linear space and $f_1, f_2, ..., f_n$ be in E^*. Let $M = \bigcap_{i=1}^{n} f_i^{-1}(0)$ and G be a subspace of finite codimension in M. The annihilator of G in M is isometrically isomorphic to the subset $P_M^{-1}(0) \cap G^{\perp}$ of E^*. Thus we have

Remark 2. If M and G are defined as above, then

$$G \overset{(p)}{\subset} M \Rightarrow \text{Every } f \in P_M^{-1}(0) \cap G^{\perp} \text{ is strongly orthogonal to the subspace } M^{\perp} \text{ of } E^*.$$

In the particular case when G is a hyperplane in M, the above is also a sufficient condition. In other words, if $G = [f_1, f_2, ..., f_n, g]$, for $g \in E^*$, then $G \subset^{(p)} M$ if g is strongly orthogonal to the subspace $M^{\perp} = [f_1, ..., f_n]$.

PROPOSITION 5. *Let E be a $P(2)$ space. Then $S(E_1)$ is orthogonally linear.*

Proof. Let f_1 and f_2 be linear functionals in $S(E_1)$ such that f_1 is strongly orthogonal to f_2. We have to show that $[f_1, f_2] \subseteq S(E_1)$.

Let $G = \bigcap_{i=1}^{2} f_i^{-1}(0)$ and $M = f_2^{-1}(0)$. Then by Remark 1, $G \subset^{(p)} M$. Further, $M \subset^{(p)} E$ since $f_2 \in S(E_1)$. Thus $G \subset^{(p)} M$, $M \subset^{(p)} E$, and $\dim E/G = 2$. But E is a $P(2)$ space and this implies that $G \subset^{(p)} E$. This in turn implies that $G^{\perp} = [f_1, f_2] \subseteq S(E_1)$.

We are now in a position to give a characterization of $P(n)$ spaces.

THEOREM 6. *Let E be a normed linear space and n be a positive integer $\geqslant 2$. Then E is a $P(n)$ space if and only if E is a $R(n, 1)$ space and $S(E_1)$ is orthogonally linear.*

Proof: *Necessity.* A $P(n)$ $(n \geqslant 2)$ space is also a $P(2)$ space, and by Proposition 4, $S(E_1)$ is orthogonally linear.

We will now show that E is a $R(n, 1)$ space. To begin with, we prove that E is a $R(2, 1)$ space and then use induction to show that E is a $R(n, 1)$ space.

Assume that G is a subspace of codim 2 in E with $G^\perp \subseteq S(E_1)$. Let $f_1 \in G^\perp$ be chosen arbitrarily. Since $f_1 \in S(E_1)$, there exists $x \in E_1$ such that $f_1(x) = \|f_1\|$. Select $f_2 \in G^\perp$ such that $f_2(x) = 0$. Then f_1 is strongly orthogonal to f_2. We have $G = \bigcap_{i=1}^{2} f_i^{-1}(0)$. Taking $M = f_2^{-1}(0)$, we see that $G \subset^{(p)} M$ by Remark 1. Also $M \subset^{(p)} E$ since $f_2 \in S(E_1)$. Thus $G \subset^{(p)} M$, $M \subset^{(p)} E$, and dim $E/G = 2$. Since E is a $P(2)$ space, this implies $G \subset^{(p)} E$, and hence E is a $R(2, 1)$ space.

Now assume inductively that E is a $R(k-1, 1)$ $(3 \leqslant k \leqslant n)$ space. We will show that E is a $R(k, 1)$ space. Let G be any subspace of codimension k in E with $G^\perp \subset S(E_1)$. We claim that $G \subset^{(p)} E$.

To show this let $g \in G^\perp$ be an arbitrary functional. Since $g \in S(E_1)$, there exists $x \in E_1$ such that $g(x) = \|g\|$. Construct a basis $g, f_1, f_2, ..., f_{k-1}$ of $G^\perp$ such that $f_i(x) = 0$, for $1 \leqslant i \leqslant k-1$. Define $M = \bigcap_{i=1}^{k-1} f_i^{-1}(0)$. Then $M^\perp = [f, ..., f_{k-1}]$ and codim $M = k-1$. Since $M^\perp \subset G^\perp \subseteq S(E_1)$ and E is assumed to be a $R(k-1, 1)$ space, we have $M \subset^{(p)} E$. Now G is a hyperplane in M and g is strongly orthogonal to $M^\perp$. Hence by Remark 1, $G \subset^{(p)} M$. Thus $G \subset^{(p)} M$, $M \subset^{(p)} E$, and, further, dim $E/G = k \leqslant n$. Since E is a $P(n)$ space, this implies $G \subset^{(p)} E$.

Sufficiency. Let E be a $R(n, 1)$ space with $S(E_1)$ orthogonally linear. Let $G \subset^{(p)} M$, $M \subset^{(p)} E$ with dim $E/G \leqslant n$. We have to show that $G \subset^{(p)} E$. Since E is a $R(n, 1)$ space, it suffices to show that $G^\perp \subseteq S(E_1)$.

The space $M^\perp$ is proximinal in $G^\perp$ and thus $G^\perp = (P_{M^\perp}^{-1}(0) \cap G^\perp) + M^\perp$. We have $M \subset^{(p)} E$ and this implies that $M^\perp \subseteq S(E_1)$. Also $G \subset^{(p)} M$, and by Remark 1, we have $P_{M^\perp}^{-1}(0) \cap G^\perp \subseteq S(E_1)$ and each functional in $P_{M^\perp}^{-1}(0) \cap G^\perp$ is strongly orthogonal to $M^\perp$. Since $S(E_1)$ is orthogonally linear, this implies that $G^\perp \subseteq S(E_1)$.

COROLLARY 5. *A normed linear space is a P space if and only if it is a $R(1)$ space and $S(E_1)$ is orthogonally linear.*

COROLLARY 6. *The space c_0 and the incomplete inner product spaces are P spaces.*

COROLLARY 7. *The infinite dimensional* $C(Q)$ *and* $L_1(T, v)$ *spaces are not* $P(2)$ *spaces and hence are not* $P(n)$ *spaces for any* $n \geqslant 2$.

4. CHARACTERIZATION OF $R(n, 1)$ SPACES

In this section we give two characterizations of $R(n, 1)$ spaces. It does not seem easy to identify new $R(n, 1)$ spaces using these results. However, any characterization of $R(n, 1)$ spaces would involve finite dimensional subspaces of $S(E_1)$ (of which little seems known) and therefore is likely to have the same defect in some sense. We feel that sufficient conditions, like the one given by Corollary 1, would be more useful in identifying $R(n, 1)$ spaces.

For F, a linear subspace of E^*, let τ_F denote the weak topology, defined on E, generated by the elements of F. Thus a net $\{x_\alpha\}$ in E converges to x in the τ_F-topology if and only if $x^*(x_\alpha) \to x^*(x)$, for all $x^* \in F$. Then we have the following lemma.

LEMMA 7. *Let* M *be a subspace of finite codimension in* E. *Then* M *is proximinal in* E *if and only if* E_1 *is* $\tau_{M^\perp}$*-compact.*

Proof. We will show that

$$E_1 \text{ is } \tau_{M^\perp}\text{-compact} \Leftrightarrow \theta_M(E_1) = (M^\perp)_1^*.$$

This, in conjunction with Theorem A, would imply the conclusion of the lemma.

Let (x_α) be a net in E_1. Then for $x \in E_1$, we have

$$(x_\alpha) \to x \text{ in the } \tau_{M^\perp}\text{-topology} \Leftrightarrow \theta_M(x_\alpha) \to \theta_M(x) \text{ in } (M^\perp)_1^*.$$

Hence

$$\begin{aligned} E_1 \text{ is } \tau_{M^\perp}\text{-compact} &\Leftrightarrow \theta_M(E_1) \text{ is compact in } (M^\perp)_1^* \\ &\Leftrightarrow \theta_M(E_1) \text{ is closed in } (M^\perp)^*, \end{aligned} \tag{5}$$

since $\theta_M(E_1)$ is bounded and $(M^\perp)^*$ is finite dimensional. Now $\hat{E}_1$ is w^*-dense in E_1^{**}, and so $\theta_M(E_1)$ is dense in $(M^\perp)_1^*$. This together with (5) implies that E_1 is $\tau_{M^\perp}$-compact $\Leftrightarrow \theta_M(E_1) = (M^\perp)_1^*$.

THEOREM 8. *A normed linear space* E *is a* $R(n, 1)$ *space if and only if* E_1 *is* τ_F*-compact for every finite dimensional subspace* F *of* $S(E_1)$ *with* $\dim F \leqslant n$.

Proof. If F is a subspace of E^*, let

$$F_{\perp}=\{x\in E: f(x)=0, \text{ for every } f\in F\}.$$

Then it is easy to verify that E is a $R(n, 1)$ space if and only if $F_{\perp}$ is proximinal in E for every finite dimensional subspace F of $S(E_1)$ with $\dim F\leqslant n$. Now the conclusion of the theorem follows from Lemma 7.

COROLLARY 8. *A normed linear space E is a $R(1)$ space if and only if E_1 is τ_F-compact for every finite dimensional subspace F of $S(E_1)$.*

If C is any closed convex subset of a normed linear space E, let Ext(C) *denote the extreme points of C. Then we have*

THEOREM 9. *A normed linear space E is a $R(n, 1)$ space if and only if*

$$\text{Ext}(E_1^{**})\subseteq \hat{E}_1+F^{\perp},$$

for every finite dimensional subspace F of $S(E_1)$ with $\dim F\leqslant n$.

Proof: Necessity. Assume that E is a $R(n, 1)$ space. Let F be a k-dimensional $(1\leqslant k\leqslant n)$ subspace of $S(E_1)$. Then $M=F_{\perp}$ is proximinal. Further, $F=(F_{\perp})^{\perp}=M^{\perp}$. Hence, by Theorem A, $\theta_M(E_1)=(M^{\perp})_1^*=F_1^*$.

Let $\Phi_0\in \text{Ext}(E_1^{**})$. Let Φ be the restriction of Φ_0 to the subspace F of E^*. Then $\Phi\in F_1^*$. Hence there exists $x\in E_1$ with $\theta(x)=\Phi$. This implies that $\Phi_0-\hat{x}\in F^{\perp}$ or, equivalently, $\Phi_0\in \hat{x}+F^{\perp}$.

Sufficiency. Let M be a subspace of codimension k where $1\leqslant k\leqslant n$ with $M^{\perp}\subseteq S(E_1)$. We have to show that $M\subset^{(P)} E$ or, equivalently, by Theorem A, $\theta_M(E_1)=(M^{\perp})_1^*$.

We have $(M^{\perp})_1^*=$ Convex hull of $\text{Ext}((M^{\perp})_1^*)$, since $(M^{\perp})^*$ is a finite dimensional space. Also $\theta_M(E_1)$ is a convex subset of $(M^{\perp})_1^*$, and therefore it suffices to show that $\text{Ext}((M^{\perp})_1^*)\subseteq\theta_M(E_1)$ to prove our claim.

Let $\Phi\in\text{Ext}((M^{\perp})_1^*)$ and Φ_0 be an extremal extension of Φ to E_1^{**}. Then $\Phi_0\in\text{Ext}(E_1^{**})$, and by our assumption there is an $x\in E_1$ such that $\Phi_0\in\hat{x}+M^{\perp}$. This implies that $\Phi\in\hat{x}+M^{\perp}$. Hence $\Phi(f)=f(x)$, for every $f\in M^{\perp}$. Also $\|\Phi\|=\|x\|=1$, since $x\in E_1$ and Φ attains its norm on $M^{\perp}$. Thus $\theta_M(x)=\Phi$. Since $\Phi\in\text{Ext}((M^{\perp})_1^*)$ was chosen arbitrarily, this proves our claim.

Remark 3. Let F be any finite dimensional subspace of $S(E_1)$. The space $F^{\perp}$ is a w^* closed subspace of E^{**} and hence is proximinal. Thus $E^{**}=P_{F^{\perp}}^{-1}(0)+F^{\perp}$. What is needed further to satisfy the condition of Theorem 9 is that

$$\text{Ext}(E_1^{**})\subseteq(P_{F^{\perp}}^{-1}(0)\cap\hat{E})+F^{\perp}. \tag{6}$$

We note that for x in E, if $\hat{x} \in P_{F^\perp}^{-1}(0)$, then

$$\|x\| = \|\hat{x}\| = \inf_{\Phi \in F^\perp} \|\hat{x} - \Phi\| = \sup_{f \in F_1} f(x),$$

and therefore (6) implies the condition of Theorem 9.

5. Linear Structure of $S(E_1)$

For a normed linear space to be a $P(n)$ space, by Theorem 6, involves two conditions, viz., E must be a $R(n, 1)$ space and $S(E_1)$ must be orthogonally linear. The only known examples of spaces E for which $S(E_1)$ is orthogonally linear are c_0 and the inner product spaces. In both the cases $S(E_1)$ is, in fact, a linear subspace of E^*. We do not know of any space E for which $S(E_1)$ is not linear but is orthogonally linear.

Very little seems to be known about the linear structure of the set $S(E_1)$. In this section we make a few observations about the orthogonal linearity and linearity of $S(E_1)$ and pose some questions.

Let M be a subspace of a normed linear space E and $f \in E^*$. We have

$$S(M_1) = \{f \in E^* : \exists x \in M_1 \ni f(x) = \sup_{y \in M_1} f(y)\}$$
$$= \{f \in E^* : \exists x \in M_1 \ni f(x) = \|f \mid M\|\}.$$

Proposition 10. *Let E be a normed linear space. Then $S(E_1)$ is orthogonally linear if and only if $S(H_1) \subseteq S(E_1)$, for every proximinal hyperplane H in E.*

Proof: Necessity. Assume that $S(E_1)$ is orthogonally linear. Let H be any proximinal hyperplane in E. Then $H = h^{-1}(0)$, for some linear functional $h \in S(E_1)$. Let $f \in S(H_1)$. Consider $f \mid H$. Let f_0 be a norm preserving extension of $f|_H$ to E^*. Then $f = f_0 + \alpha h$, for some scalar α, and f_0 attains its norm on E at a point in H. Thus f_0 is strongly orthogonal to $h \in S(E_1)$. Since $S(E_1)$ is orthogonally linear, this implies that $f \in S(E_1)$.

Sufficiency. Assume that $S(H_1) \subseteq S(E_1)$, for every proximinal hyperplane H in E.

Let f_1 and f_2 in $S(E_1)$ be such that f_1 is strongly orthogonal to f_2. We claim that $[f_1, f_2] \subseteq S(E_1)$. Let $H = f_2^{-1}(0)$. Then H is a proximinal hyperplane in E and $f_1 \in S(H_1)$. Clearly, if g is any functional in the subspace $[f_1, f_2]$, $g \in S(H_1)$. This implies that $g \in S(E_1)$, and the claim is proved.

PROPOSITION 11. *Let E be a normed linear space. Then $S(E_1)$ is linear if $S(H_1)=S(E_1)$, for every proximinal hyperplane H in E.*

Proof. By Proposition 10, $S(E_1)$ is orthogonally linear. To show that $S(E_1)$ is linear, consider any f_1 and f_2 in $S(E_1)$. Let $H=f_2^{-1}(0)$. Then H is a proximinal hyperplane and $f_1 \in S(H_1)$. Let f_0 be a norm preserving extension of $f_1 | H$ to E^*. Then $f_0 \in S(E_1)$, and f_0 is strongly orthogonal to f_2. Since $S(E_1)$ is orthogonally linear, we have $[f_0, f_2] \subseteq S(E_1)$. But $f_1 - f_0 \in [f_2]$, and so $[f_0, f_2] = [f_1, f_2]$. Hence $[f_1, f_2] \subseteq S(E_1)$.

COROLLARY 9. *Let E be a $R(2, 1)$ space. Then $S(E_1)$ is linear if and only if $S(H_1)=S(E_1)$, for every proximinal hyperplane H in E.*

Proof. We have to prove only the necessity. Let E be a $R(2, 1)$ space with $S(E_1)$ linear. Clearly, $S(E_1)$ is orthogonally linear, and so by Proposition 10, $S(H_1) \subseteq S(E_1)$, for every proximinal hyperplane H in E.

To prove the other inclusion, consider any $f \in S(E_1)$ and a proximinal hyperplane H in E. Then $H=h^{-1}(0)$, for some $h \in S(E_1)$. Since $S(E_1)$ is linear, we have $[f, h] \subseteq S(E_1)$. Further, E is a $R(2, 1)$ space and so $[f, h]_\perp \subset^{(P)} E$. This implies that $[f, h]_\perp$ is a proximinal hyperplane in H. Hence $f \in S(H_1)$. Since f and H were arbitrarily chosen, this proves our claim.

Now we list some questions for which we do not have an answer.

Problems. (1) If E is a normed linear space, does orthogonal linearity of $S(E_1)$ imply linearity of $S(E_1)$?

(2) Are there any P spaces other than c_0 and the inner product spaces?

(3) Are there any nonreflexive normed linear spaces E, other than c_0 and the inner product spaces for which $S(E_1)$ is linear or orthogonally linear?

(4) We observe that all the known examples of $R(2, 1)$ spaces are, in fact, $R(1)$ spaces. Is there a normed linear space E which is a $R(2, 1)$ space but not a $R(n, 1)$ space for $n \geqslant 3$?

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